MATH 54 – MIDTERM 2

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Name: ______________________________________

Instructions: This midterm counts for 20% of your grade. You officially have 110 minutes to take this exam. Good luck, and don't worry, you'll be fine!

1		10
2		20
3		10
4		10
5		10
6		10
7		5
8		10
9		8
10		5
11		2
Bonus		4
Total		100

Date: Friday, July 13th, 2012.

1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) If A is a $m \times n$ matrix, then $dim(Nul(A)) + Rank(A) = m$

(b) The change-of-coordinates matrix P from $\mathcal{B}$ to $\mathcal{C}$ has the property that $[\mathbf{x}]_{\mathcal{B}} = P\,[\mathbf{x}]_{\mathcal{C}}$

(c) If $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then $Nul(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(d) The set of polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(3) = 0$ is a subspace of P_2

(e) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

(a)	
(b)	
(c)	
(d)	
(e)	

2. *(20 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

(a) The set of matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

(b) The matrix of the linear transformation T which reflects points in $\mathbb{R}^2$ about the x-axis and then about the y-axis is the same as the matrix of the linear transformation S which rotates points in $\mathbb{R}^2$ about the origin by 180 degrees counterclockwise.

(c) The following set is a basis for P_2: $\{1, 1+t, 1+t+t^2\}$

(d) If V is a set that contains the $\mathbf{0}$-vector, and such that whenever $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V, then V is a vector space!

Note: Read the instructions on the previous page, they will be taken more seriously this time!

3. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Row(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. *(10 points)* Let $\mathcal{B} = \{7 - 2t, 2 - t\}$, and $\mathcal{C} = \{4 + t, 5 + 2t\}$ be bases for P_1.

   Calculate $[\mathbf{x}]_{\mathcal{C}}$ given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

   Hint: First calculate a change-of-coordinates matrix!

5. *(10 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$$

Find the matrix of T relative to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } M_{2\times 2}$$

6. *(10 points)* Define $T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

(a) *(5 points)* Show that T is a linear transformation

(b) *(5 points)* Find a basis for $Ker(T)$ (or $Nul(T)$ if you wish).
Show that the basis you found is in fact a basis (i.e. is linearly independent and spans $Ker(T)$)!

Note: This is a tiny bit longer than you think!

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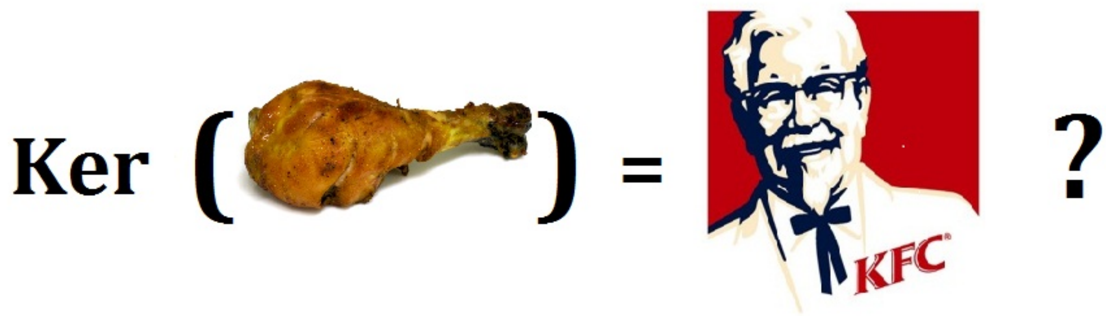

Note: If you don't get it, the right-hand-side is Colonel Sanders (which sounds like Kernel) :)

7. *(5 points)* Find the largest *open* interval (a, b) on which the following differential equation has a unique solution:

$$(x-3)y'' + \left(\sqrt{x}\right) y' = \sqrt{x-1}$$

with

$$y(2) = 3, y'(2) = 1$$

8. *(10 points)* Solve the following differential equation:

$$y''' - 3y'' + 12y' - 10y = 0$$

9. *(8 points)* Solve the following differential equation:

$$y'' - 3y' + 2y = e^{3t}$$

(a) *(4 points)* Using undetermined coefficients

(b) *(4 points)* Using variation of parameters

10. *(5 points)*

(a) *(1 point)* If $T : V \to W$ is a one-to-one linear transformation and $T(\mathbf{x}) = \mathbf{0}$, what can you say about $\mathbf{x}$?

(b) *(4 points)* Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly *independent* vectors (in V) and $T : V \to W$ is a *one-to-one* linear transformation. Show that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are also linearly independent.

Hint: Use (a)!!!

Let me start the proof for you:

Suppose $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$.

We want to show that $a = b = c = 0$

Note: You can actually use this problem to show that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ and A is an invertible matrix, then $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is also a basis for $\mathbb{R}^3$. This provides a neat way of creating new bases for $\mathbb{R}^3$: Start with any old basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{R}^3$, then $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is a (usually) a new basis for $\mathbb{R}^3$.

11. *(2 points)* What is the basis of your happiness? :)

Bonus. *(4 points)* In this problem, we're going to use the Wronskian to find the general solution of a quite complicated differential equation! This should illustrate yet again the power of the Wronskian!

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian (determinant):

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

By differentiating $W(t)$ with respect to t, find a simple differential equation satisfied by $W(t)$ and solve it.

Note: You answer will involve the $\int$ sign!

Note: From now on, ignore the constants, i.e. in your answer in (a), set $C = 1$

(b) From (a), we get:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \underline{\hspace{4cm}} \text{(your answer from } (a))$$

Solve for y_2 in terms of y_1.

Hint: Divide this equality by $(y_1(t))^2$ and recognize the left-hand-side as the derivative of a quotient, and hence solve for y_2 in terms of y_1. You answer will involve another $\int$ sign!

Note: Again, ignore the constants!

(c) Let's apply the result in (b) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $\boxed{y_1(t) = \sin(t)}$.

Use your answer in (b) to find *another* solution $y_2(t)$!

Note: Again, you can ignore the constants!

Hint: You should use the following facts, in order:

1) $\int \tan(t)dt = -\ln(\cos(t))$
2) At some point, multiply your integrand (the fct you're integrating) by $\frac{\cos(t)}{\cos(t)}$
3) The substitution $u = \frac{1}{\sin(t)}$
4) The formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$
5) The function $\coth^{-1}(z) = \frac{1}{2}\ln\left|\frac{1-z}{1+z}\right|$
6) Try to have a formula *without* $\frac{1}{\sin(t)}$ (see how you can simplify the stuff inside the $\ln$)

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the general solution of $y'' - \tan(t)y' + 2y = 0$? (no need to show your work here)

(Scratch work)

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Any comments about this exam? (too long? too hard?)